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In this research project we have established several landmark results in large deviation theory, invariant properties of common statistical tests, Bahadur efficiency computations and the study of standby redundant systems. These results have numerous applications to Statistical methodology, Monte carlo simulations, bootstrap methods and Bayesian statistics and in the study of reliability of systems.

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Large Deviation Limit Theorems, With Applications

FINAL REPORT

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1. Summary Of Research

1.1 Statement of the problems studied.

In this research project we have established several landmark results in large deviation theory, invariant properties of common statistical tests, Bahadur efficiency computations and the study of standby redundant systems. These results have numerous applications to Statistical methodology, Monte carlo simulations, bootstrap methods and Bayesian statistics and in the study of reliability of systems. In the next four sections below we outline briefly the technical details of significant accomplishments of the work done under this contract.

1.2 Summary of the most important results.

1.2.1 Large deviation theory.

Let Ω be a Polish space, that is, a complete separable metric space and $\mathcal{B}$ be the Borel σ-field on Ω containing all the open and closed subsets of Ω. A function $I(x) : \Omega \to [0, \infty]$ is said to be a *rate function* if it is lower semi-continuous. Let $\{\mu_n\}$ be a sequence of probability measures on $(\Omega, \mathcal{B})$. We say that $\{\mu_n\}$ obeys *large deviation principle* (LDP) with rate function $I(x)$ if the following conditions are satisfied:

$$(1) \qquad \limsup_n \frac{1}{n} \log \mu_n(C) \le -I(C)$$

$$(2) \qquad \liminf_n \frac{1}{n} \log \mu_n(G) \ge -I(G)$$

for all closed sets C and for all open sets G of Ω. The rate function $I(x)$ is known as a *proper rate function* if for each $L \ge 0$, the level set $\{x : I(x) \le L\}$ is a compact subset of

Ω. Note that proper rate functions are also rate functions, since a nonnegative function is lower semi-continuous if and only if the level sets are closed.

Let $(\Omega_1, \mathcal{B}_1)$, $(\Omega_2, \mathcal{B}_2)$ be two Polish spaces with their associated Borel σ-fields. Let $\{\mu_{1n}\}$ be a sequence of probability measures on $(\Omega_1, \mathcal{B}_1)$ and $\{\nu_n(x_1, B_2)\}$ be a sequence of transition functions on $\Omega_1 \times \mathcal{B}_2$. Consider a sequence of probability measures $\{\mu_n\}$ on the product space $(\Omega, \mathcal{B}) = (\Omega_1 \times \Omega_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ given by

$$\mu_n(B_1 \times B_2) = \int_{B_1} \nu_n(x_1, B_2) \, d\mu_{1n}(x_1)$$

for $B_i \in \mathcal{B}_i$, $i = 1, 2$.

We say that the sequence of probability transition functions $\{\nu_n(x_1, \cdot), x_1 \in \Omega_1\}$ satisfies the LDP *continuously in* x_1 with rate function $J(x_1, x_2)$, or simply LDP *continuity condition* holds, if

(i) For each $x_1 \in \Omega_1$, $J(x_1, \cdot)$ is a proper rate function on Ω_2.

(ii) For any sequence $\{x_{1n}\}$ in Ω_1 such that $x_{1n} \to x_1$, the sequence of measures $\{\nu_n(x_{1n}, \cdot)\}$ on Ω_2 obeys the LDP with rate function $J(x_1, \cdot)$, and

(iii) $J(x_1, x_2)$ is jointly lower semi-continuous in (x_1, x_2).

Suppose that the sequence $\{\mu_{1n}\}$ obeys the LDP with proper rate function $I_1(x_1)$ and the sequence of probability transition functions $\{\nu_n(x_1, \cdot), x_1 \in \Omega_1\}$ satisfies the LDP continuously in x_1 with rate function $J(x_1, x_2)$. In this research project we have shown that under these conditions, the sequence of joint distributions $\{\mu_n\}$ obeys the LDP with rate function $I(x_1, x_2) = I_1(x_1) + J(x_1, x_2)$. There are several interesting applications of this theorem in statistics. For example, we use our theorem to show that the joint distribution of the ordinary empirical measure of a sample and the corresponding bootstrap empirical measure obeys the LDP in the weak topology. Other applications include large deviations for empirical Bayes methods and random sums.

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1.2.2 Invariant properties of common statistical tests.

The common common statistical tests are derived under the assumption that samples are taken independently from one or more normal populations. Several authors have studied the effect of nonnormality on these tests. Typically, this part of the robustness literature makes the assumption of mutual independence of the observations. While the independence assumption may be approximately valid due to the choice of experimental designs, clearly the case of dependence between the observations is of practical as well as aesthetic interest. One can even argue that in practical applications, observations are frequently not iid and the physical systems responsible for generation of the observations automatically introduce some dependence among the observations. It is then clear that the statistical interest should be to determine if procedures valid under the independence assumption continue to remain valid with only a simple adjustment when independence assumption is violated.

In this research project we have obtained interesting characterizations of covariance structures under which the usual procedures remain valid with possibly a scale factor adjustment. This was accomplished first by obtaining a characterization of all nonnegative (n.n.d.) solutions Σ of a consistent matrix equation $A \Sigma A = B$, where A is any symmetric matrix and B is any n.n.d. matrix. This result is of independent interest, and is useful in characterization problems concerning generalized inverses of matrices. We have used our result to characterize the class of covariance matrices such that the distributions of several both univariate and multivariate test statistics remain invariant.

1.2.3 Bahadur efficiency.

The most important application of large deviation theory in statistics is the calculation of Bahadur slopes and Bahadur efficiencies. In fact the notion of Bahadur slope

provided impetus for the development of large deviation theory. Both the establishment of the LDP for a sequence of distributions and the identification of the rate function are essential for explicit calculations of Bahadur slopes. We can describe briefly the concept of Bahadur slope as follows. Let $X_1, \ldots, X_n$ be iid $f_\theta(x)$, where $\theta \in \Omega$. Suppose we want to test the hypothesis

$$H : \theta = \theta_0 \quad vs \quad K : \theta \in \Omega_1.$$

Let $T_n = T_n(X_1, \ldots, X_n)$ be a real valued test statistic with distribution $F_{n\theta}$ and consider the test which rejects the null hypothesis H for large values of T_n. Given an observed value of $T_n = t_n$, the p-value or the level of significance is given by

$$p_n = P_{\theta_0}(T_n > t_n) = [1 - F_{n\theta_0}(t_n)].$$

Motivated by this Bahadur (1960) defined the level attained by T_n as the random variable

$$L_n = 1 - F_{n\theta_0}(T_n).$$

Usually, under the null hypothesis, L_n is uniformly distributed over $(0, 1)$ and under the alternative $L_n \to 0$ exponentially fast. We shall say that the sequence $\{T_n\}$ has exact slope $c(\theta)$ if

$$\lim_n \frac{-2}{n} \log L_n = c(\theta)$$

almost surely under θ. Note that $c(\theta)$ determines the rate at which $L_n \to 0$ as $n \to \infty$ and the theory of large deviations is normally used to determine $c(\theta)$. Now suppose that we have two test statistics T_{1n} and T_{2n} with slopes $c_1(\theta)$ and $c_2(\theta)$. If $c_1(\theta) > c_2(\theta)$ and θ is the true value then the test based on T_{1n} becomes significant sooner than the test based on T_{2n}, that is, T_{1n} requires much smaller sample size to reject the null hypothesis then T_{2n}. Thus a measure of asymptotic relative efficiency of T_{1n} relative to T_{2n} known as Bahadur efficiency is given by the ratio $c_1(\theta)/c_2(\theta)$.

In this research project as an important application of our theorem concerning LDP for joint distributions, we have derived Bahadur slopes of test statistics for the contaminated normal model, which is useful to study robustness of these test statistics. There

are several other potential applications, which include Bahadur slopes for UMPU tests for asymmetric rejection regions, and the study of Bahadur efficiencies in the presence of nuisance parameters.

1.2.4 Redundant systems.

The increasing need for highly available and reliable computing systems, such as for military, space, and commercial applications, the study of methods to build such systems has become necessary. In general, high availability and reliability can be achieved through replication of components and/or systems. Based on the type of redundancy and the usage of the redundant components, the redundant computing systems are classified into two major types

(1) *Standby redundant systems* execute tasks only on their active modules. Upon detection of failure of an active module, these systems attempt to replace the failed unit with a spare. Thus, as in the previous case, there is no computational advantage due to redundancy. The system's availability,however, is increased.

(2) *Gracefully degrading systems* may use all units to execute tasks, i.e., all failure-free units are active. Hence, the redundancy is also used to obtain computational gains via parallelization of task execution. When failure is detected in a unit, these systems attempt to reconfigure to a system with one fewer unit.

These systems, both the standby redundant and the gracefully degrading systems are frequently employed to obtain high system reliability through hardware redundancy. In this research project we have studied performance analysis for gracefully degrading systems as well as for standby redundant systems with special emphasis on the exponential case.

1.3   List of all publications.

1.3.1   Papers published.

1. Strong Large Deviation and Local Limit Theorems, (with J. Sethuraman).
   Annals of Probability, Vol. 21, No. 3, pp1671-1690, 1993.

2. Order Statistics Based Modeling of Gracefully Degrading Computing Systems.
   (with R. Mukkamala). J. Microelectronics and Reliability, Vol. 33, No. 9, pp
   1281-1291, 1993.

3. On "A matrix formulation on how deviant an observation can be" by Olkin.
   The American Statistician, Vol 47, No. 2, p158, 1993.

4. Modeling and Analysis of Standby Redundant Computing Systems.
   (with R. Mukkamala). J. Microelectronics and Reliability, Vol. 34, No. 2, pp
   323-334, 1994.

5. On "A matrix formulation on how deviant an observation can be", Rejoinder to
   appear in The American Statistician, November 1994.

1.3.2   Papers submitted for publication.

1. Inequalities for positive semidefinite matrices and statistical applications, (with
   Akhil Vaish).

2. An invariance property of common statistical tests, (with Akhil Vaish).

3. Multidimensional strong large deviation theorems. (with J. Sethuraman).

4. Large deviations for the bootstrap empirical measure, (with R. L. Karandikar).

5. Large deviations for joint distributions and statistical applications.

1.3.3   Papers under preparation.

1. Bahadur slopes of common test statistics for scale mixture of normal distributions.

2. Wishartness and independence of quadratic forms under special covariance structures, (with Akhil Vaish).

3. Wishartness and independence of quadratic forms under general covariance structure, (with Akhil Vaish).

1.4   Ph. D. Dissertations.

1. Invariance Properties of Statistical Tests for Dependent Observations, Akhil Vaish.

1.5   Presentations and Invited Talks.

1. Large deviations for the bootstrap, Dept. of Statistics, Univ. of North Carolina, Chapel Hill, June 1992.

2. Large deviations for the bootstrap empirical measure, Annual meeting of the IMS, SanFrancisco, August 1993.

3. Large deviations for joint distributions and statistical applications, Workshop in large deviations, Stanford University, August 1993.

4. Bahadur slopes of common test statistics for contaminated normal distribution,

   Dept. of Statistics, Purdue University, W. Lafayette, April 1993.

   Dept. of Mathematics, Oakland University, Detroit, July 1993.

   Dept. of Statistics, University of Connecticut, Storrs, October 1993.

   Dept. of Statistics, Pennsylvania State University, State College, November 1993.

   Dept. of Math. and Stat., University of Maryland-Baltimore County, April 1994.

5. Large deviation principle for common test statistics, IMS-Bernoulli Society meetings, Chapel Hill, NC, June 1994.

6. An invariance property of common statistical tests. Annual Meetings, ASA, Toronto, August 1994.

1.6 Participating Scientific Personnel.

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